THE PROBLEM OF THE ELECTRIC FIELD OF AN ELECTRODE WITH A PRE-ELECTRODE POTENTIAL DROP IN A MEDIUM WITH TENSOR CONDUCTIVITY

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We investigate the nonlinear current distribution in an electrode of finite dimensions with a pre-electrode layer in which the potential locally depends on the current density. The electrode is in contact with a medium of anisotropic conductivity caused by the Hall effect. The problem is reduced to the solution of a nonlinear integrodifferential equation. It is shown that the structure of the field is determined by the Hall parameter $\omega\tau$ and the form of the volt-ampere characteristic in the pre-electrode layer.

From theoretical considerations based on idealized assumptions about the properties of conductors, dielectrics, and the media surrounding them, the current distribution in electrodes of finite dimensions becomes significantly nonuniform with singularities at the end points. This nonuniformity in the current, as already remarked on several occasions, is amplified in media in which the Hall effect appears. At the present time all the fundamental relationships have been obtained in the approximate theory of three-dimensional fields in the flows of ionized gases and in semiconductors. The consequences which have been deduced from theory have been confirmed by experimental verification [1, 2].

However, in specific cases, in comparing theoretical calculations with experimental results, there are divergences indicating limitations in the application of the approximate theory [3]. It may be remarked that there are more important physical phenomena which are not accounted for in the original equations and not reflected in the boundary conditions of linear theory problems, but which, evidently have a significant effect on the formation of the field. These phenomena include the nonlinear conductivity of the medium in strong electric fields, which is observed, for example, in an unbalanced plasma and in semiconductors with "hot electrodes," and also contact phenomena at the boundary of heterogeneous media. To take account of these new factors, defining the nonlinear properties of fields, as a rule makes the solution of the boundary value problems extremely complicated. Nevertheless, it is necessary for an analysis of the processes.

1. Within the framework of the phenomenological theory of a continuous medium we consider the effect of a potential drop in the pre-electrode layer on the current distribution in an electrode adjacent to the flow of an anisotropic conducting plasma.

We use the theoretical description of the phenomena in the pre-electrode layer proposed by Lyubimov and Vatazhin and used by them to compute, in linear approximation, the two-dimensional fields in magnetogasdynamic channels with scalar conducting flows [4, 5]. In this theory two fundamental assumptions are made:

1) The thickness of the layer is small by comparison with a typical length in the problem.

2) The potential drop in the layer, φ_*, locally depends on the normal component of the current.

The form of the function φ_* is defined by the physicochemical properties of the plasma, the material of the electrode, and is established by theoretical calculations or taken from experiments. In the large the pre-electrode processes relate to the surface of the electrode and are taken into account in the effective

Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 1, pp. 19-23, January-February, 1971. Original article submitted October 12, 1970.

boundary conditions $\varphi = \varphi_0 + \varphi_* (j_n)$, where φ_0 is the actual potential of the electrode. In other words, the normal component of the current density vector at the surface matches some additional potential due to the purely local value of the field and the predetermined boundary properties of the plasma and the electrode surface.

We assume that the electrode of finite dimensions $ab = 2l$, $-l \le x \le l$ for $y = 0$ (the remaining part of the x axis is a dielectric) is in contact with a two-dimensional flow of an incompressible anisotropic conducting medium $\mathbf{v}\,[u(x, y), v(x, y), 0]$ filling the lower half-plane. The external magnetic field $\mathbf{H}(0, 0, H_z)$ is assumed to be everywhere homogeneous and to considerably exceed the intrinsic field of the currents to be determined, the effect of the latter field being ignored.

The fundamental equations of the theory of the electric field in this case

$$j_x(x, y) = \sigma_{xx}(H)\left(-\frac{\partial\varphi}{\partial x} + \frac{1}{c}vH\right) + \sigma_{xy}(H)\left(-\frac{\partial\varphi}{\partial y} - \frac{1}{c}uH\right) \tag{1.1}$$

$$j_y(x, y) = \sigma_{yx}(H)\left(-\frac{\partial\varphi}{\partial x} + \frac{1}{c}vH\right) + \sigma_{yy}(H)\left(-\frac{\partial\varphi}{\partial y} - \frac{1}{c}uH\right)$$

$$\frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} = 0 \qquad \left(\begin{matrix}\sigma_{xx}(H) = \sigma_{yy}(H), & \text{for electrons}\\ \sigma_{xy}(H) = -\sigma_{yx}(H), & \sigma_{xx}(H) > 0\\ & \sigma_{xy}(H) < 0\end{matrix}\right)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

have to be solved with the boundary conditions

$$\varphi = \varphi_0 + \varphi_*(j_y) \qquad -l < x < l \quad \text{for} \quad y = 0$$

$$j_y(x) = 0 \quad |x| > l \quad \text{for} \quad y = 0, \qquad \mathbf{j} \to 0 \quad \text{for} \quad (x, y) \to \infty \tag{1.2}$$

$$\int_{-l}^{l} j_y(x, 0)\,dx = I, \quad \mathrm{v} = 0 \quad \text{for} \quad y = 0$$

In the above we have used commonly accepted notation.

In the boundary conditions we have assumed that $\varphi_*(j_y)$ and the total current I passing through the electrode are given. Other electrodes through which the current is closed are assumed to be infinitely distant. The components of the electrical conductivity tensor σ_{xx}, σ_{xy} in (1.1) depend only on the magnetic field.

In (1.1) we can introduce the complex current which can be put in the form of a Cauchy type integral

$$j(z) = j_x(x, y) - ij_y(x, y) = \frac{1}{\pi}\int_{-l}^{l}\frac{j_x(x)\,dx}{x - z} \qquad (z = x + iy)$$

$$(-l < x < l, \quad \operatorname{Im} z < 0) \tag{1.3}$$

This satisfies the boundary conditions (1.2) and, for large $|z|$, has the expansion

$$j(z) = -\frac{z^{-1}}{\pi}\int_{-l}^{l} j_y(x)\,dx - \frac{z^{-2}}{\pi}\int_{-l}^{l} xj_y(x)\,dx - \ldots = -\frac{I}{\pi z} + O(z^{-2}) \tag{1.4}$$

We differentiate the first boundary condition of (1.2) with respect to x

$$\frac{d\varphi(x)}{dx} - \frac{d\varphi_*(j_y)}{dj_y}\frac{dj_y(x)}{dx} = 0 \qquad (-l < x < l, \quad y = 0) \tag{1.5}$$

and write (1.5), using the first two equations of (1.1), as

$$\beta(H)\,j_y(x) + j_x(x) + \sigma(H)\,F(j_y)\frac{dj_y(x)}{dx} = 0, \qquad F(j_y) = \frac{d\varphi_*(j_y)}{dj_y} \tag{1.6}$$

Here the electrical conductivity $\sigma(H)$ and the Hall parameter $\beta(H)$ in the magnetic field are defined by the equations

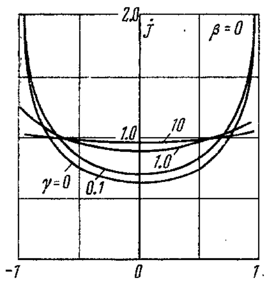

Fig. 1

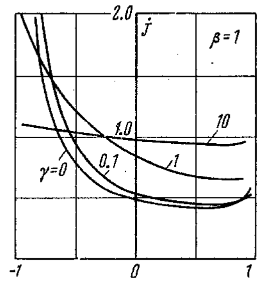

Fig. 2

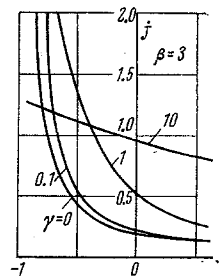

Fig. 3

$$\sigma(H)=\frac{\sigma_{xx}^2(H)+\sigma_{xy}^2(H)}{\sigma_{xx}(H)},\quad R(H)=\frac{\sigma_{xy}(H)}{H[\sigma_{xx}^2(H)+\sigma_{xy}^2(H)]} \tag{1.7}$$
$$\beta(H)=\sigma(H)R(H)H$$

If we substitute the boundary value of the integral (1.3) in ab for $j_x(x)$ in (1.6) we obtain a nonlinear singular integrodifferential equation for the normal component of the current density vector at the electrode

$$\beta(H)j_y(H)+\frac{1}{\pi}\int_{-l}^{l}\frac{j_y(t)\,dt}{t-x}+\sigma(H)F(j_y)\frac{dj_y(x)}{dx}=0\quad(-l<x,\ t<l) \tag{1.8}$$

where β and σ are constant for fixed H. The pre-electrode phenomena in (1.8) correspond to the third term with coefficient $F(j_y) = d\varphi_*/dj_y$, the form of which specifies the volt-ampere characteristic $\varphi_*(j_y)$.

2. We have not been able to construct analytic solutions for Eq. (1.8). However, the properties of the fields described by it can be established by analyzing particular solutions which are obtained by numerical methods.

First we note the case when the potential in the pre-electrode layer is independent of the current, i.e., when $\varphi_* =$ const, $F = 0$. Then Eq. (1.8) becomes a linear singular integral equation with a closed contour of integration and its solution, obtained by solving the appropriate Riemann boundary value problem, has the form [6, 7]

$$\begin{aligned}
j_x(x)&=\frac{\beta(H)I}{\sqrt{1+\beta^2(H)}}(l+x)^{-1/2-\varepsilon}(l-x)^{-1/2+\varepsilon}\\
&\qquad\qquad\qquad\qquad(-l<x<l)\\
j_y(x)&=\frac{-I}{\sqrt{1+\beta^2(H)}}(l+x)^{-1/2-\varepsilon}(l-x)^{-1/2+\varepsilon}\\
\varepsilon&=\pi^{-1}\operatorname{arc\,tg}\beta\qquad(0\leqslant\varepsilon<1/2)
\end{aligned} \tag{2.1}$$

i.e., is the familiar solution of the problem of the current distribution in an ideal electrode when the electrical conductivity of the surrounding medium is anisotropic.

Turning to the numerical solution of Eq. (1.7), we integrate it once

$$\beta(H)\int_0^x j_y(x)\,dx-\frac{1}{\pi}\int_{-l}^{l}j_y(t)\ln|t-x|\,dt+\sigma(H)\int_0^{j_y}F(j_y)\,dj_y=C=\text{const} \tag{2.2}$$

and assume that the volt-ampere characteristic is given by the function $\varphi_*=aj_y+b$, where a and b are constants ($F = a$). We can reduce Eq. (2.2) to a system of algebraic equations by the method of finite differences, which we can write in nondimensional form ($j^0 = j_y l/I$, $x^0 = x/l$; the superscripts are omitted from the equations which follow)

$$\begin{aligned}
&\frac{\beta}{n}\sum_{p=1}^{p}j_p-\frac{1}{\pi}\sum_{k=1}^{2n}j_k\Big(\frac{2k-2p+1}{2n}\ln\frac{|2k-2p+1|}{2n}-\frac{2k-2p-1}{2n}\\
&\times\ln\frac{|2k-2p-1|}{2n}-\frac{1}{n}\Big)+\gamma j_p=C,\quad\frac{1}{n}\sum_{k=1}^{2n}j_k=2\qquad(\gamma=a\sigma(H))
\end{aligned} \tag{2.3}$$

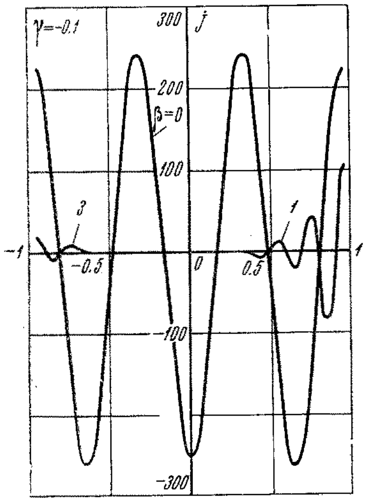

Fig. 4

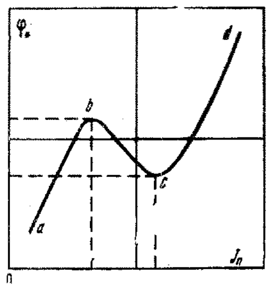

Fig. 5

This last equation corresponds to the integral condition

$$\int_{-l}^{l} j_y(x)\,dx = I$$

in (1.2) and it must be included in the system for (2.3) to be determinate, since after integrating (1.8), the right side of (2.2) contains the unknown constant C. At the points

$$\xi = -1 + \frac{2p-1}{2n}, \quad -1 + \frac{k-1}{n} < \xi < \frac{k}{n} - 1 \quad (p,\, k = 1, \ldots, 2n) \qquad (2.4)$$

Eqs. (2.3) exactly satisfy Eq. (1.8) or the equivalent Eq. (2.2).

The results of computations on a computer for three values of the Hall parameter $\beta(H) = 0, 1, 3$ and four values of the nondimensional parameter $\gamma = a\,\sigma(H) = 0, 0.1, 1, 10$ ($a > 0$) are given in Figs. 1, 2, and 3. It follows from the form of the curves that there is a common property for all values of β – the distribution of the normal component of the current density along the length of the electrode flattens out as γ increases. This is explained as follows. At points where the current gradient is nonzero ($dj_y/dx \neq 0$), large values of j_y in the pre-electrode layer correspond to large values of the potential φ_*, which limits the increase in j_y, since we have assumed that the volt-ampere characteristic increases. As a result of the compatibility between φ_* and j_y the current distribution at the electrode is reorganized, j_y decreases when φ_* is large and, conversely, increases where φ_* is small; the inhomogeneity in $j_y(x)$ decreases and in general vanishes as $\gamma \to \infty$ ($dj_y/dx \to 0$). This property of the field occurs to a lesser extent as β increases and σ decreases, the latter because $\gamma = a\sigma$ (a = const).

The current distribution is quite different for falling volt-ampere characteristic ($a < 0$). Here increase in j_y leads to a reduction in φ_* and the nonuniformity of current flow at the electrode must increase. However, it is difficult to predict the form of the function $j_y(x)$. In this case the solution of Eqs. (2.3) depends on the relations between β, a, and σ and is sensitive to changes in them, small variations in the coefficients of Eq. (1.8) leading to large changes in the current density which can become of variable sign at the electrode. In this sense we can discuss the instability of the distribution of $j_y(x)$. As an example, Fig. 4 shows the results of solving the equation for three values of $\beta(H) = 0, 1, 3$ and $\gamma = -0.1$. The graphs show that because of the change in the sign of $j_y(x)$ there are current vortices near the electrode, the form of which strongly depends on the size of the external magnetic field or, more precisely, on the value of the Hall parameter $\beta(H)$.

If the volt-ampere characteristic $\varphi_*(j_y)$ has increasing and decreasing parts (for example, has N-shaped form; Fig. 5) the current distribution has properties characteristic of the two cases discussed above for $a > 0$ and $a < 0$.

It is possible that the occurrence of falling parts in the volt-ampere characteristic of the pre-electrode layer is associated with the formation of arcs and patches which have been observed experimentally at the electrode, and also with the appearance of current and voltage fluctuations in the load.

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